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1996 J. Phys. A: Math. Gen. 29 7797

(http://iopscience.iop.org/0305-4470/29/23/035)

J. Phys. A: Math. Gen. **29** (1996) 7797–7801. Printed in the UK

A quantum model with one bosonic degree of freedom

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Received 20 March 1996, in final form 25 June 1996

Abstract. Conventionally, when we construct a quantum model, we must first know the corresponding classical model. Applying the correspondence between the classical Poisson brackets and the canonical commutator, we can find the canonical quantization condition. Through the example given in this paper, we will find that it is not necessary to do this. In the example, a Lagrangian operator is operator gauge invariant. After localization, in order to keep the operator gauge invariance of the operator action, we must introduce a gauge potential. The Euler–Lagrange equation of motion of q gives the usual operator equation of motion, and the gauge potential B_0 gives a constraint. This constraint is just the usual canonical quantization condition.

In quantum mechanics and quantum field theory, in order to obtain a canonical commutation relation, we need to know the corresponding classical model, since in the formalism of quantum theory, the canonical commutation relation cannot be obtained automatically. So, we must analyse the related classical model first, and try to deduce the Poisson brackets between the canonical coordinate and the corresponding conjugated momentum. Then applying the correspondence between the classical Poisson brackets and the canonical commutation relation, we can find the canonical commutation relation. Therefore, in this case, the canonical quantization condition is a quantum hypothesis. But can we obtain it from a more fundamental principle?

This work was first done by Adler [1, 2], who developed a generalized quantum dynamics. He used the concepts of operator-valued gauge transformations and a total trace action—concepts which he introduced [1, 2, 3–5]; operator-valued gauge transformations were also discussed by Mackey [6, 7]. Adler's formalism gives the usual operator equations of motion, with the canonical commutation relations emerging as constraints with the operator gauge potential [1, 2]. His formalism is different from the usual quantum mechanics and quantum field theory.

In this paper, we explore a quantum model with one bosonic degree of freedom. The starting point of all discussions is that all the physical arguments, such as q, Lagrangian, Hamiltonian and action etc, are operator-valued variables. In this model, the conventional canonical quantization condition emerges as a constraint of the operator gauge potential. This means that the conventional canonical quantization condition is automatically contained in the structure of the model. So we need not use the conventional canonical procedure of 'quantizing' a related classical mechanics to obtain a quantum mechanical model. The goal

0305-4470/96/237797+05$19.50

of our constructing this model is to discuss how to spontaneously introduce the canonical quantization condition in the formalism of the usual quantum theory.

Now we discuss a quantum harmonic oscillator. First, we simply discuss a model with global unitary symmetry. Its Lagrangian is

$$L = \frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{1}{2}\omega^2 q^2 \tag{1}$$

where q and its time derivative $\mathrm{d}q/\mathrm{d}t$ are quantum variables, and ω is a c-number parameter. We can easily prove that the above Lagrangian is invariant under the following transformation (later we will prove it):

$$q \to q' = Uq \tag{2}$$

where the operator U is unitary:

$$UU^+ = 1 = U^+U\,. \tag{3}$$

Now, we localize the theory. In gauge theory, when a gauge transformation is localized, in order to ensure the gauge invariance, we must introduce a gauge field. In the present model, in order to ensure unitary invariance, we will introduce a gauge potential B_0. Then the Lagrangian is changed into

$$L = \tfrac{1}{2}(\mathrm{D}_0 q)^2 - \tfrac{1}{2}\omega^2 q^2 - \tfrac{1}{2}\mathrm{i}B_0 \tag{4}$$

where D_0 is a covariant derivative, whose definition is

$$\mathrm{D}_0 q = (\partial_t + B_0)q = \frac{\mathrm{d}q}{\mathrm{d}t} + B_0 q\,. \tag{5}$$

From equation (4), equation (5) is the canonical momentum conjugate to the coordinate q.

The corresponding operator action is

$$S = \int_{-\infty}^{\infty} L\,\mathrm{d}t\,. \tag{6}$$

We will prove that the operator action is invariant under the following transformations:

$$q \to q' = U(t)q \tag{7}$$

$$\mathrm{D}_0 q \to \mathrm{D}'_0 q' = U(t)\mathrm{D}_0 q \tag{8}$$

$$B_0 \to B'_0 = U(t)B_0U^+(t) - \frac{\mathrm{d}U(t)}{\mathrm{d}t}U^+(t) \tag{9}$$

where $U(t)$ is a unitary operator:

$$U(t)U^+(t) = 1 = U^+(t)U(t)\,. \tag{10}$$

Obviously, when we let $U(t)$ be independent of time t and the gauge potential B_0 equal zero, the localized Lagrangian (4) is the same as the original Lagrangian (1), and the present theory should be able to return to the original one. So a natural requirement is that the dynamical variables in the present theory have similar properties to the corresponding dynamical variables in the original theory. For example, in the original theory, $q, \mathrm{d}q/\mathrm{d}t$ and $\mathrm{d}^2q/\mathrm{d}t^2$ are Hermitian operators, so we require that $q, \mathrm{D}_0 q$ and $\mathrm{D}_0\mathrm{D}_0 q$ be Hermitian operators. The requirement of the Hermiticity of $q', \mathrm{D}'_0 q', L, \mathrm{D}_0 q$ and $\mathrm{D}_0\mathrm{D}_0 q$ gives the following five restrictions:

$$Uq = qU^+ \tag{11}$$

$$U(\mathrm{D}_0 q) = (\mathrm{D}_0 q)U^+ \tag{12}$$

$$B_0^+ = -B_0 \tag{13}$$

$$\{q, B_0\} = 0 \tag{14}$$

$$\{D_0 q, B_0\} = 0 \tag{15}$$

respectively, where $U = U(t)$ and the braces represent a conventional anti-commutator.

Applying the properties (11) and (12) of U, it is easy to understand that the transformations (7) and (8) have the form of operator gauge transformations. Resolve U into the square of another operator V. V must have the same properties of (11) and (12) as that of U. Therefore, equations (7) and (8), respectively, change into:

$$q \to q' = V^2 q = VqV^+ \tag{16}$$

$$D_0 q \to D_0' q' = V(D_0 q)V^+ \, . \tag{17}$$

These are just the form of operator gauge transformations [1, 2, 6, 7].

Now we discuss the change of the Lagrangian L under the transformations defined by (7)–(9). Using relations (10)–(12), we can prove that q^2 and $(D_0 q)^2$ are invariant:

$$q^2 \to q'^2 = qU^+Uq = q^2 \, . \tag{18}$$

Similarly,

$$(D_0 q)^2 \to (D_0' q')^2 = (D_0 q)^2 \, . \tag{19}$$

Therefore, the change of the Lagrangian L under the transformations is

$$\delta L = -\tfrac{1}{2} \mathrm{i} \delta B_0 \, . \tag{20}$$

In order to prove the unitary invariance of the operator action, we discuss an infinitesimal operator-valued transformation of the form

$$U = 1 + \delta\Lambda \qquad \delta\Lambda = -\delta\Lambda^+ \, . \tag{21}$$

Using equations (11) and (12), we found that $\delta\Lambda$ must satisfy the following restrictions:

$$\{q, \delta\Lambda\} = 0 \tag{22}$$

$$\{D_0 q, \delta\Lambda\} = 0 \, . \tag{23}$$

From equations (21)–(23), we find that $\delta\Lambda$ satisfies the same algebra as that of B_0 which is defined by equations (13)–(15). Now, we try to find the relationship between U and B_0; we look at it from a mathematical point of view. We know that (12) holds in this model for any coordinate q, any permissive gauge potential B_0 and any permissive transformation operator U. In particular, it should hold when $\dot{q}$ vanishes. When $\dot{q}$ vanishes, according to (5) equation (12) changes into the following form:

$$UB_0 q = B_0 q U^+ \, . \tag{24a}$$

According to (11), and noticing that q is any coordinate variable, equation (24*a*) then gives the following relation:

$$[U, B_0] = 0 \, . \tag{24b}$$

Using equation (21), we found that $\delta\Lambda$ commutes with B_0:

$$[\delta\Lambda, B_0] = 0 \, . \tag{24c}$$

Then the first-order variation of B_0 is

$$\delta B_0 = -\frac{\mathrm{d}(\delta\Lambda)}{\mathrm{d}t} \, . \tag{25}$$

Under an infinitesimal transformation, the first-order variation of the operator action S is

$$\delta S = \mathrm{i}\tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\delta\Lambda\,. \tag{26}$$

So when $\delta\Lambda$ vanishes at $t = \pm\infty$, or more generally, $\delta\Lambda(\infty) = \delta\Lambda(-\infty)$, the operator action is invariant, i.e.

$$\delta S = 0 \tag{27}$$

which means that our model has local operator gauge symmetry.

Now we discuss the Euler–Lagrange equation of motion. First we consider the equation of motion of q. Let q change infinitesimally arbitrarily, meanwhile, δq vanishes at $t = \pm\infty$, the first-order variation of the operator action is

$$\delta S = \int_{-\infty}^{\infty} \mathrm{d}t\,\tfrac{1}{2}\{\mathrm{D}_0\mathrm{D}_0 q + \omega^2 q, \delta q\}\,. \tag{28}$$

Because δq is an arbitrary variation of q, we can let δq be proportional to the unit operator in the Hilbert space. Therefore, the Hamilton action principle gives the following equation of motion:

$$\mathrm{D}_0\mathrm{D}_0 q + \omega^2 q = 0 \tag{29}$$

which is just the usual operator equation of motion.

If we let B_0 change infinitesimally arbitrarily, using relation (14) and (15), the first-order variation of the operator action is

$$\delta S = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t[q(\mathrm{D}_0 q) - (\mathrm{D}_0 q)q - \mathrm{i}]\delta B_0\,. \tag{30}$$

So the Hamilton action principle gives the following constraint:

$$[q, \mathrm{D}_0 q] = \mathrm{i} \tag{31}$$

where $\mathrm{D}_0 q$ in (31) is the conjugated momentum conjugate to the coordinate q. Thus equation (31) is just the usual canonical commutation relation.

Finally, we simply discuss the transformation operator U and the gauge potential B_0 [9]. From Strocchi and Wightman's work [8], we know that this transformation is very restrictive. So is B_0, but we can find a solution to them [9]. Using equations (21), (11) and (12), we found that $\delta\Lambda$ satisfies the same algebra as that of B_0, which is defined by (13)–(15). A possible solution of B_0 and $\delta\Lambda$ has the following form:

$$\mathrm{e}^{\pi(\mathrm{D}_0 q)q}\,. \tag{32}$$

Using relation (31), we can easily prove this. Therefore, the transformation operator U and the gauge potential B_0 that satisfy all the restrictions exist. In this model, an operator gauge potential is just a constraint. It is not an independent dynamical argument. So B_0 could be expressed by q and $\dot{q}$. Therefore we can let B_0 be proportional to (32). From equations (22) and (23), we found that $\delta\Lambda$ commutes with B_0 which is consistent with our preceding result (equation (24*c*)).

From the above discussions, we know that the canonical quantization condition can be introduced spontaneously through the symmetry of the Hilbert space, and, in some sense, the hypothesis about the symmetry of the Hilbert space is more fundamental and more natural, and is thus very important in quantum theory.

Acknowledgments

This work is supported in part by the foundation of PhD directing programme of the Chinese University, Professor T D Lee's NNSC grant, the foundation of National Natural Science of China and grant LWTV/1298 of the Chinese Academy of Science.

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